

Toward a Unified Theory of Switching:

Constraint Geometry, Hierarchical Reorganization, and Gait Transitions

Akira Minegishi

Minegishi Akira Consulting Office
Switching Science Project
Tokyo, Japan

ORCID: 0009-0005-6420-340X

Abstract

Switching between behavioral modes is a ubiquitous phenomenon observed across biological, cognitive, and organizational systems. Despite extensive research on locomotor transitions, motor coordination, and adaptive reorganization, no general theory currently explains switching across domains. Existing accounts remain fragmented among dynamical systems theory, hierarchical organization models, constraint-based optimization, and motor variability analysis.

Building on four preceding studies of locomotor switching, this paper demonstrates that four seemingly distinct frameworks, SCAN, the Viable System Model (VSM), Constraint Geometry, and the 3D Minimal Model, share a common underlying structure. Despite differences in terminology and mathematical formulation, each framework describes switching as a five-stage process involving: (1) stabilization within a current mode, (2) destabilization and expansion of degrees of freedom, (3) traversal of a boundary (liminal) region, (4) transition along a minimal pathway, and (5) re-stabilization in a new mode.

Based on this structural equivalence, we propose the **Unified Switching Theory**, a domain-general framework centered on three core principles: (1) an increase in task-irrelevant variability ($\text{Var}\perp$), (2) traversal of a boundary region in which multiple modes become accessible, and (3) geodesic transition toward a new stable configuration. Within this framework, task-irrelevant variability serves as a common indicator of destabilization, while geodesic transitions provide a general principle governing mode reorganization.

We show that the proposed theory accounts not only for the walking-running transition but also for switching phenomena in motor coordination, skill acquisition, organizational restructuring, and cognitive mode changes. By integrating structural, geometric, and dynamical perspectives, the Unified Switching Theory provides a common language for understanding how complex systems reorganize, adapt, and transition between modes. The theory offers a foundation for future mechanistic, quantitative, and cross-domain investigations of switching phenomena.

Keywords:

Switching Theory; Gait Transition; Constraint Geometry; Task-Irrelevant Variability; SCAN; Geodesic Transition; Hierarchical Reorganization; Complex Systems

Chapter 1. Introduction

Switching between distinct modes of behavior is a ubiquitous phenomenon observed across biological, cognitive, and organizational systems. Human locomotion provides a particularly clear example: as walking speed increases, humans spontaneously transition from walking to running at a characteristic critical speed. Although this transition has been studied for decades, existing explanations remain

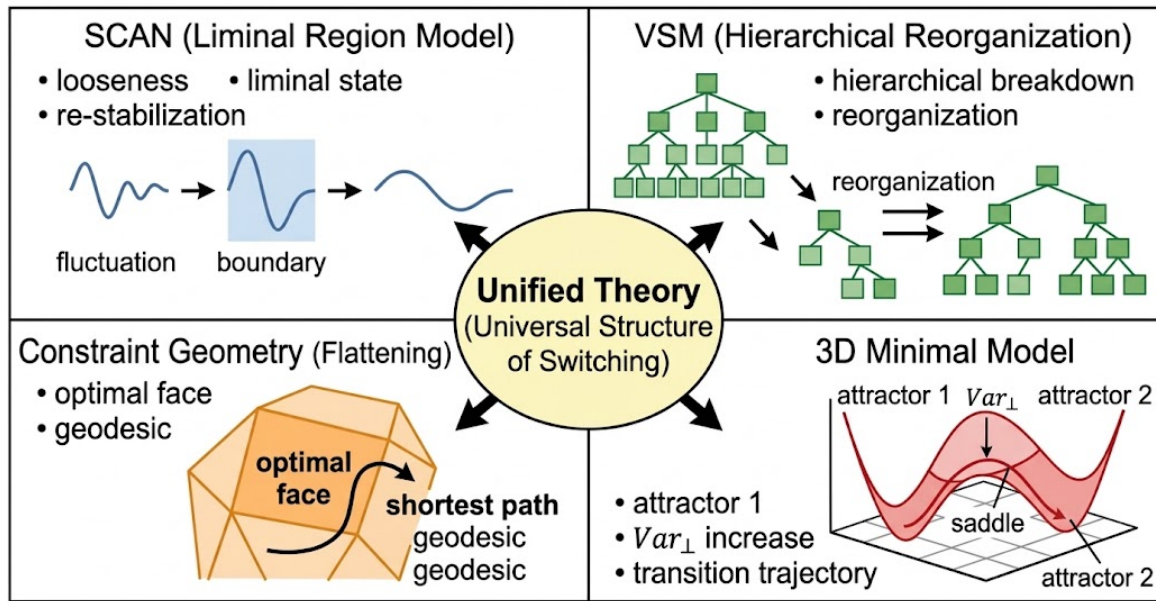


Figure 1: Overview of the Unified Switching Theory. SCAN, VSM, Constraint Geometry, and the 3D Minimal Model are complementary representations of a common switching structure. SCAN characterizes looseness, liminality, and re-stabilization; VSM emphasizes hierarchical breakdown and reorganization; Constraint Geometry represents movement across an optimal face along a geodesic; and the 3D Minimal Model describes attractor shallowing, increased task-irrelevant variability ($Var_{\perp}$), and transition trajectories. The Unified Switching Theory integrates these perspectives through freedom increase, boundary-region traversal, and shortest-path transition toward a new stable mode.

fragmented across multiple theoretical traditions—ranging from dynamical systems theory and motor control to hierarchical organization theory and constraint-based optimization. Each framework captures an important aspect of the switching process, yet none alone provides a comprehensive account of *why* switching occurs, *how* it unfolds, and what structural principles govern it across domains.

In four preceding studies on locomotor switching, we developed and analyzed four complementary frameworks:

1. **SCAN (looseness–liminal–re-stabilization)** — highlighting the progressive loss of tight control, entry into a boundary region, and emergence of a new stable mode.
2. **VSM (hierarchical breakdown and reorganization)** — interpreting switching as a restructuring of functional hierarchies under increasing task demands.
3. **Constraint Geometry (optimal face and geodesic transitions)** — revealing that switching corresponds to movement across an expanded optimal face along a shortest path.
4. **3D Minimal Model (attractor shallowing, $\text{Var}\perp$ increase, and transition trajectories)** — providing a dynamical systems account grounded in a minimal three-dimensional state space.

Despite their differences, these four frameworks exhibit a striking structural convergence. Each describes switching as a process involving (i) destabilization of the current mode, (ii) expansion of available degrees of freedom, (iii) traversal of a boundary region, and (iv) emergence of a new stable configuration. This convergence suggests the existence of a deeper, domain-general structure underlying switching phenomena.

Beyond the traditions of motor control and dynamical systems theory, the present perspective is also broadly consistent with classical analyses of feasible structures, constraint-induced adaptation, and organizational reconfiguration developed in mathematical economics and organization theory (Arrow, 1974; Nikaido, 1960; Aoki, 1971).

The purpose of this paper is to formalize this deeper structure as a unified theoretical framework—the *Unified Switching Theory*. Our goal is not merely to compare the four frameworks, but to demonstrate that they are structurally equivalent representations of the same underlying process. Building on this equivalence, we derive a general theory that explains switching as a universal pattern of:

- increasing freedom along task-irrelevant dimensions,
- boundary-region traversal, and
- geodesic transition toward a new stable state.

This unified theory provides a principled account of why switching occurs, how it unfolds, and why its structure is preserved across domains as diverse as human locomotion, motor coordination, organizational restructuring, and cognitive mode shifts.

The contributions of this paper are threefold:

1. **Structural Integration** We demonstrate the formal equivalence among SCAN, VSM, Constraint Geometry, and the 3D Minimal Model, showing that each framework captures the same five-stage switching structure.

2. **Unified Switching Theory** We introduce a general theoretical formulation that synthesizes these perspectives into a single coherent framework, centered on the role of $Var\perp$ (task-irrelevant variability) and geodesic transitions.
3. **Cross-Domain Applicability** We show that the unified theory extends beyond locomotion to other domains, including motor redundancy (UCM hypothesis), organizational mode switching, and bipedal stance transitions.

By integrating structural, geometric, and dynamical perspectives, this paper aims to provide a comprehensive and generalizable account of switching phenomena. The unified theory not only clarifies the mechanisms underlying the walking–running transition but also offers a conceptual foundation for understanding switching across a wide range of natural and artificial systems.

Chapter 2. Preliminaries

This chapter summarizes the four theoretical frameworks that form the foundation of the unified switching theory developed in this paper. The goal is not to provide exhaustive reviews—each framework has been treated in detail in Papers 1–4—but rather to formalize the essential structures required for the integration presented in Chapters 3 and 4. Throughout this chapter, we adopt a common notation that highlights the shared elements across frameworks, particularly the distinction between task-relevant (TR) and task-irrelevant (TI) dimensions, and the role of variability along these dimensions.

2.1 SCAN: Looseness, Liminality, and Re-stabilization

SCAN conceptualizes switching as a three-stage process:

1. **Looseness** A gradual reduction in the tightness of control within the current mode. Variability increases, particularly along TI dimensions, reflecting a weakening of the existing attractor.
2. **Liminality** A boundary region in which the system is no longer fully committed to the current mode but has not yet transitioned to the next. Control becomes distributed, and multiple potential modes coexist.
3. **Re-stabilization** Emergence of a new stable mode with renewed tightness and reduced variability.

Formally, let $x \in \mathbb{R}^n$ denote the system state, and let $\sigma(x)$ represent the tightness of control. SCAN characterizes switching as a trajectory in which:

$$\sigma(x) \downarrow (\text{looseness}), x \in \mathcal{L}(\text{liminal region}), \sigma(x) \uparrow (\text{re-stabilization}).$$

The liminal region $\mathcal{L}$ plays a central role in the unified theory.

2.2 VSM: Hierarchical Breakdown and Reorganization

Originally proposed for organizational systems, VSM is used here as a general framework for hierarchical control architectures.

The Viable System Model (VSM) interprets switching as a restructuring of functional hierarchies. A hierarchical organization H maintains stability under normal conditions, but increasing task demands induce:

1. **Hierarchical Breakdown** Loss of coherence in the existing hierarchy. Functional units become less tightly coupled, and control becomes decentralized.
2. **Reorganization** Emergence of a new hierarchy H' better suited to the new task demands.

Let $\mathcal{C}(H)$ denote the coherence of hierarchy H . Switching corresponds to:

$$\mathcal{C}(H) \downarrow (\text{breakdown}), H \rightarrow H' (\text{reorganization}).$$

This breakdown–reorganization structure parallels SCAN’s looseness–liminal–re-stabilization sequence.

2.3 Constraint Geometry: Optimal Faces and Geodesic Transitions

Constraint Geometry provides a static, optimization-based perspective on switching. A system operating under constraints can be represented by a feasible polytope $P \subset \mathbb{R}^n$. Within this polytope, the system occupies an **optimal face** $F \subseteq P$ determined by task demands.

As task demands change, the optimal face expands:

$$F \subset F' \subset F'' \subset \dots$$

Switching occurs when the system moves from one optimal face to another along a **geodesic**—the shortest path within the feasible region.

Key concepts:

- **Optimal face**: region of feasible solutions achieving the same optimal value.
- **Face expansion**: increased degrees of freedom as constraints loosen.
- **Geodesic transition**: minimal-distance movement between faces.

This geometric structure aligns with the dynamical transitions observed in the 3D minimal model.

2.4 The 3D Minimal Model: Attractors, $\text{Var}\perp$, and Transition Trajectories

The 3D minimal model provides a dynamical systems representation of switching in a reduced state space:

$$x = (x_{TR}, x_{TI}, v),$$

where:

- x_{TR} : task-relevant dimension
- x_{TI} : task-irrelevant dimension
- v : control parameter (e.g., speed)

The model exhibits:

1. **Attractor Shallowing** As v increases, the walking attractor becomes shallower.
2. **Increase in $\text{Var}\perp$** Variability along the TI dimension increases:

$$Var_{\perp} = Var(x_{TI}) \uparrow$$

3. **Transition Trajectory** The system follows a saddle-connecting trajectory toward the running attractor.

This model provides the dynamical foundation for the unified theory, linking variability, attractor geometry, and transition pathways.

2.5 Summary

The four frameworks—SCAN, VSM, Constraint Geometry, and the 3D Minimal Model—share a common structural pattern:

- destabilization of the current mode,
- expansion of available degrees of freedom,
- traversal of a boundary or transitional region,
- emergence of a new stable configuration.

These shared elements form the basis for the structural equivalence demonstrated in Chapter 3 and the unified theory developed in Chapter 4.

Chapter 3. Structural Equivalence of Four Frameworks

The four frameworks reviewed in Chapter 2—SCAN, VSM, Constraint Geometry, and the 3D Minimal Model—were originally developed in distinct research traditions and for different purposes. Despite their differences in terminology, mathematical formulation, and conceptual emphasis, they exhibit a remarkable structural convergence when applied to switching phenomena. This chapter demonstrates that each framework describes the same underlying five-stage process, thereby establishing the foundation for the unified switching theory developed in Chapter 4.

3.1 Extraction of the Five-Stage Switching Structure

Across the four frameworks, switching consistently appears as a process involving the following five stages:

1. **Stable Mode** The system operates within a well-defined attractor, hierarchy, or optimal face.
2. **Looseness / Destabilization** Stability weakens; variability increases, especially along task-irrelevant dimensions.
3. **Boundary Region (Liminality)** The system enters a transitional region where multiple modes coexist or become accessible.
4. **Transition Pathway** The system moves along a minimal or geodesic path toward a new configuration.
5. **Re-stabilization** A new stable mode emerges with renewed tightness and reduced variability.

These stages provide a common template for analyzing switching across frameworks.

3.2 Correspondence with SCAN

SCAN explicitly encodes the first three stages:

- **Stable Mode** → **Looseness** → **Liminality** → **Re-stabilization**

Looseness corresponds to the early destabilization of the current mode, while the liminal region represents the boundary region in which the system is no longer committed to a single mode. Re-stabilization corresponds to the emergence of a new attractor.

SCAN does not explicitly model the transition pathway, but the liminal region implicitly contains the conditions under which the transition occurs.

3.3 Correspondence with VSM

VSM describes switching as:

- **Stable Hierarchy → Breakdown → Reorganization → New Hierarchy**

The breakdown of hierarchical coherence corresponds to looseness in SCAN. The period of distributed control and ambiguity corresponds to the liminal region. Reorganization corresponds to the transition pathway and re-stabilization.

Thus, VSM provides a structural and functional interpretation of the same five-stage process.

3.4 Correspondence with Constraint Geometry

Constraint Geometry provides a geometric interpretation:

- **Optimal Face → Face Expansion → Interior Region → Geodesic Transition → New Face**

The expansion of the optimal face corresponds to looseness and increased degrees of freedom. The interior region of the expanded face corresponds to the liminal region. The geodesic transition corresponds to the transition pathway. Arrival at a new optimal face corresponds to re-stabilization.

This framework makes explicit the shortest-path nature of the transition.

3.5 Correspondence with the 3D Minimal Model

The 3D minimal model provides a dynamical systems interpretation:

- **Stable Attractor → Attractor Shallowing → Var $\perp$ Increase → Saddle-Connecting Trajectory → New Attractor**

Attractor shallowing corresponds to looseness. Increased Var $\perp$ corresponds to the expansion of degrees of freedom. The saddle region corresponds to the liminal region. The transition trajectory corresponds to the geodesic pathway. The new attractor corresponds to re-stabilization.

This model provides the dynamical foundation for the unified theory.

3.6 Summary of Structural Equivalence

Table 1 summarizes the correspondence among the four frameworks.

Table 1. Structural correspondence among the four switching frameworks.

Stage	SCAN	VSM	Constraint Geometry	3D Minimal Model
Stable Mode	tight	coherent hierarchy	optimal face	deep attractor
Looseness / Destabilization	looseness	breakdown	face expansion	attractor shallowing
Boundary Region	liminal	distributed control	interior region	saddle region

Stage	SCAN	VSM	Constraint Geometry	3D Minimal Model
Transition Pathway	—	reorganization	geodesic	transition trajectory
Re-stabilization	new tight	new hierarchy	new face	new attractor

These correspondences demonstrate that the four frameworks are structurally equivalent representations of the same underlying switching process. This structural equivalence provides the conceptual and mathematical foundation for the unified switching theory developed in Chapter 4.

The significance of this structural equivalence extends beyond a simple comparison of frameworks. If SCAN, VSM, Constraint Geometry, and the 3D Minimal Model describe the same underlying switching process, then they should not be viewed as competing explanations but as complementary representations of a common structure.

This observation has two important implications. First, it suggests that a unified theory of switching is possible because the four frameworks share the same fundamental organization. Second, it implies that insights developed within one domain may be transferable to others. For example, concepts originally formulated in motor control may provide useful perspectives for organizational restructuring, while geometric principles derived from constraint analysis may help explain transitions in cognitive systems.

The existence of a common switching structure therefore motivates the development of a domain-general framework capable of integrating these perspectives. The next chapter introduces such a framework under the name Unified Switching Theory.

Figure 1 summarizes this conceptual integration. The four frameworks are not treated as competing explanations, but as complementary representations of the same switching structure from structural, hierarchical, geometric, and dynamical perspectives.

Chapter 4. Unified Switching Theory

This chapter formalizes the unified switching theory by integrating the structural correspondences demonstrated in Chapter 3. The theory provides a general account of switching phenomena across biological, cognitive, and organizational systems. It is built upon three core principles:

1. **Increase in task-irrelevant variability ($\text{Var} \perp$)**
2. **Traversal of a boundary or liminal region**
3. **Geodesic transition toward a new stable configuration**

These principles synthesize the key insights from SCAN, VSM, Constraint Geometry, and the 3D Minimal Model into a single coherent framework.

4.1 Universal Structure of Switching

Across domains, switching follows a characteristic sequence of structural changes:

1. **Stable Mode** The system resides in a well-defined attractor, hierarchy, or optimal face.
2. **Freedom Increase ($\text{Var} \perp \uparrow$)** Variability increases along task-irrelevant dimensions, weakening the current mode.

3. **Boundary-Region Traversal** The system enters a liminal region where multiple modes become accessible.
4. **Shortest-Path Transition (Geodesic)** The system moves along a minimal pathway toward a new configuration.
5. **Re-stabilization** A new stable mode emerges with renewed tightness.

This sequence is the backbone of the unified switching theory.

4.2 $\text{Var}_\perp$ as the Central Integrative Quantity

A key insight emerging from the integration of the four frameworks is that **task-irrelevant variability** ($\text{Var}_\perp$) serves as a universal indicator of impending switching.

Let the system state be decomposed as:

$$x = (x_{TR}, x_{TI})$$

where:

- x_{TR} : task-relevant dimension
- x_{TI} : task-irrelevant dimension

Define:

$$\text{Var}_\perp = \text{Var}(x_{TI})$$

The unified theory asserts:

$$\text{Var}_\perp \uparrow \implies \text{increased freedom and destabilization}$$

This increase in $\text{Var}_\perp$ corresponds to:

- **SCAN**: looseness
- **VSM**: breakdown of hierarchical coherence
- **Constraint Geometry**: expansion of the optimal face
- **3D Minimal Model**: attractor shallowing and TI-axis dispersion

Thus, $\text{Var}_\perp$ provides a common quantitative language across frameworks.

4.3 Central Principle of the Unified Switching Theory

We now formalize the central principle of the unified theory.

Principle 1 (Unified Switching Principle).

Let a system operate within a stable mode characterized by a low-variability attractor or optimal configuration. If task demands or control parameters induce a monotonic increase in task-irrelevant variability $\text{Var}_\perp$, then:

1. *The stability of the current mode decreases;*

2. *The system enters a boundary (liminal) region in which multiple modes become accessible;*
3. *The transition to a new mode occurs along a geodesic (shortest-path) trajectory within the expanded feasible region;*
4. *A new stable mode emerges with reduced variability.*

Conversely, any observed switching process satisfying these four conditions can be represented within the unified switching framework.

This principle establishes switching as a **general structural process**, independent of the specific domain or underlying mechanism.

Footnote

Although the unified switching theory often assumes a globally monotonic decrease of task unrelated variability $V_{\perp}$, this assumption is a sufficient but not necessary condition. The same qualitative conclusions—namely, exit from the liminal zone and convergence toward one of the attractor basins—also hold under weaker conditions such as local or expected monotonicity, positive average drift toward lower variability, threshold crossing behavior, or asymmetric exit probabilities from the liminal zone.

4.4 Geodesic Transitions as a General Principle

Constraint Geometry provides the clearest geometric interpretation of switching: the system moves along a **geodesic** within the feasible region.

The unified theory generalizes this idea:

- In **SCAN**, the geodesic corresponds to the minimal path through the liminal region.
- In **VSM**, it corresponds to the minimal restructuring required to form a new hierarchy.
- In the **3D Minimal Model**, it corresponds to the saddle-connecting trajectory.

Thus, the geodesic is not merely a geometric artifact but a **universal transition principle**.

Formally, let $\gamma(t)$ denote the transition trajectory. Then:

$$\gamma(t) = \arg \min_{\gamma} \int \| \dot{\gamma}(t) \| dt$$

subject to feasibility constraints.

This minimality principle explains the stereotyped nature of switching trajectories across domains.

4.5 Mathematical Properties of the Unified Theory

The unified switching theory exhibits several important mathematical properties:

(1) Stability Structure

Each mode corresponds to a local minimum of an effective potential or cost function.

(2) Continuity of Transition

The transition trajectory is continuous and differentiable except at boundary points.

(3) Mode Reorganization

The new mode is structurally related to the old mode through minimal reconfiguration.

(4) Universality

The same structural pattern applies to:

- locomotor transitions
- motor coordination
- organizational restructuring
- cognitive mode switching
- ecological and social transitions

This universality is a direct consequence of the structural equivalence demonstrated in Chapter 3.

4.6 Summary

The unified switching theory provides a general, domain-independent account of switching phenomena. It integrates structural, geometric, and dynamical perspectives into a single coherent framework centered on:

- **Var $\perp$**
- **boundary-region traversal**
- **geodesic transitions**

This theory forms the conceptual foundation for the applications presented in Chapter 5.

Chapter 5. Applications of the Unified Switching Theory

The unified switching theory developed in Chapter 4 provides a general framework for understanding switching phenomena across diverse domains. This chapter demonstrates the applicability of the theory to three representative classes of systems: human locomotion, motor coordination and skill acquisition, and organizational or cognitive mode switching. These examples illustrate the breadth and explanatory power of the unified theory.

5.1 Locomotor Switching: Walking–Running Transition

The walking–running transition is one of the most widely studied switching phenomena in human movement science. Traditional explanations have focused on energetic optimality, inverted pendulum dynamics, or neuromuscular constraints. The unified switching theory provides a complementary structural account.

5.1.1 Var $\perp$ Increase and Destabilization

As walking speed increases, variability along task-irrelevant dimensions (e.g., lateral sway, step-to-step timing fluctuations) increases:

$$Var_{\perp} \uparrow$$

This corresponds to:

- **SCAN**: looseness

- **VSM:** breakdown of hierarchical gait control
- **Constraint Geometry:** expansion of the optimal face
- **3D Minimal Model:** attractor shallowing

Thus, the early phase of the transition is characterized by a systematic weakening of the walking mode.

5.1.2 Boundary Region and Transition Trajectory

Near the preferred transition speed, the system enters a boundary region in which walking and running coexist as viable modes. The transition trajectory follows a geodesic-like path through the saddle region of the 3D minimal model.

5.1.3 Re-stabilization in the Running Mode

After the transition, variability decreases and the running attractor deepens, completing the five-stage switching sequence.

The unified theory thus provides a structural explanation for the stereotyped nature of the walking–running transition.

5.2 Motor Coordination and Skill Switching

Switching is not limited to locomotion; it also appears in fine motor coordination, posture control, and skill acquisition.

5.2.1 UCM Hypothesis and $Var_{\perp}$

The Uncontrolled Manifold (UCM) hypothesis distinguishes between task-relevant (TR) and task-irrelevant (TI) dimensions. The unified theory interprets increases in TI variability as:

$$Var_{\perp} \uparrow \implies \text{freedom increase and destabilization}$$

This aligns with:

- looseness in SCAN
- breakdown in VSM
- face expansion in Constraint Geometry

Thus, UCM provides a natural quantitative instantiation of the unified theory.

5.2.2 Switching Between Coordination Patterns

Examples include:

- bimanual coordination (in-phase $\rightarrow$ anti-phase)
- postural control strategies
- transitions in skilled actions (e.g., grasping modes)

In each case:

- TI variability increases

- the system enters a boundary region
- a geodesic-like transition occurs
- a new coordination pattern stabilizes

The unified theory captures these transitions with a single structural template.

5.3 Cognitive and Organizational Mode Switching

Switching also occurs in systems far removed from biomechanics.

5.3.1 Cognitive Mode Switching

Cognitive systems exhibit transitions such as:

- focused $\rightarrow$ exploratory modes
- automatic $\rightarrow$ controlled processing
- habitual $\rightarrow$ goal-directed behavior

These transitions also follow the unified structure:

- increased variability in internal representations
- liminal coexistence of multiple cognitive modes
- geodesic-like transition in representational space
- stabilization of a new cognitive mode

Thus, the unified theory extends naturally to cognitive science.

5.3.2 Organizational Restructuring

In organizational systems, transitions such as:

- hierarchical $\rightarrow$ matrix structures
- centralized $\rightarrow$ decentralized control
- rigid $\rightarrow$ flexible workflows

follow the same five-stage structure:

- breakdown of coherence
- increased degrees of freedom
- boundary-region ambiguity
- minimal restructuring (geodesic)
- emergence of a new stable configuration

This corresponds directly to the VSM interpretation.

5.4 Summary

The unified switching theory provides a general structural account of switching phenomena across domains. Its explanatory power derives from three core principles:

1. **Increase in task-irrelevant variability ($\text{Var}\perp$)**
2. **Traversal of a boundary or liminal region**
3. **Geodesic transition toward a new stable configuration**

These principles apply to locomotor transitions, motor coordination, cognitive mode switching, and organizational restructuring, demonstrating the broad applicability and generality of the unified theory.

Chapter 6. Discussion

The unified switching theory developed in this paper provides a structural account of switching phenomena that transcends the boundaries of specific domains, models, and theoretical traditions. This chapter discusses the broader implications of the theory, its relationship to existing approaches, its limitations, and directions for future research.

6.1 Theoretical Significance

The primary contribution of this work is the demonstration that switching phenomena across diverse systems share a common structural organization. By integrating SCAN, VSM, Constraint Geometry, and the 3D Minimal Model, we have shown that switching can be understood as a universal process involving:

- destabilization of the current mode,
- expansion of degrees of freedom ($\text{Var}\perp$ increase),
- traversal of a boundary or liminal region,
- geodesic transition, and
- re-stabilization in a new mode.

This structural template provides a unifying language for describing transitions in locomotion, motor coordination, organizational restructuring, and cognitive mode switching. The theory thus bridges traditionally separate research areas and offers a general framework for understanding mode transitions in complex systems.

6.2 Relationship to Existing Approaches

6.2.1 Dynamical Systems Theory

Traditional dynamical systems approaches emphasize attractor landscapes, bifurcations, and stability analysis. The unified theory is compatible with these ideas but extends them by:

- explicitly incorporating task-irrelevant variability ($\text{Var}\perp$),
- formalizing boundary-region traversal,
- and identifying geodesic transitions as a general principle.

Thus, the unified theory provides a structural complement to dynamical systems models.

6.2.2 Optimality and Energetic Explanations

Energetic optimality has been widely used to explain locomotor transitions. The unified theory does not contradict energetic accounts but reframes them:

- Energetic changes influence the shape of the feasible region.
- This, in turn, affects $\text{Var}_{\perp}$ and the expansion of the optimal face.
- Switching emerges as a structural consequence of these changes.

Thus, energetic explanations become one mechanism among many that shape the structural conditions for switching.

6.2.3 Hierarchical and Organizational Models

VSM and related organizational theories emphasize functional restructuring. The unified theory shows that such restructuring follows the same structural pattern as motor transitions, suggesting deep analogies between biological and organizational systems.

6.2.4 Motor Redundancy and UCM

The UCM hypothesis provides a quantitative decomposition of variability. The unified theory generalizes this decomposition into a structural principle:

$$\text{Var}_{\perp} \uparrow \implies \text{freedom increase and destabilization}$$

This connects motor redundancy to broader switching phenomena.

The unified switching theory may also be viewed from a broader structural perspective. The concepts of feasible regions, constraint relaxation, and organizational reconfiguration have long been studied in mathematical economics and organization theory. From this viewpoint, switching can be interpreted as a transition between alternative viable structures under changing constraints, extending ideas discussed by Arrow (1974), Nikaido (1960), and Aoki (1971) into the domains of motor control and behavioral dynamics.

6.3 Limitations of the Unified Theory

Despite its generality, the unified theory has several limitations:

(1) Descriptive Rather Than Mechanistic

The theory identifies structural patterns but does not specify the mechanistic details that produce them in each domain.

(2) Dependence on TR/TI Decomposition

The theory assumes a meaningful decomposition into task-relevant and task-irrelevant dimensions. In some systems, this decomposition may be ambiguous or context-dependent.

(3) Lack of Quantitative Thresholds

The theory does not specify precise thresholds for $\text{Var}_{\perp}$ or stability loss. These thresholds are system-specific and require empirical determination.

(4) Limited Treatment of Stochasticity

Although variability plays a central role, the theory does not explicitly model stochastic processes or noise-driven transitions.

These limitations point to important directions for future refinement.

6.4 Future Directions

The unified switching theory opens several promising avenues for future research:

(1) Quantitative Models of $\text{Var}\perp$ Dynamics

Developing models that predict $\text{Var}\perp$ trajectories under changing task demands would strengthen the theory’s predictive power.

(2) Experimental Validation Across Domains

Empirical studies in locomotion, motor coordination, organizational behavior, and cognitive science can test the generality of the five-stage structure.

(3) Integration with Machine Learning and Robotics

Robotic systems that switch between control modes may benefit from structural principles such as geodesic transitions and $\text{Var}\perp$ -based indicators.

(4) Extension to Multi-Agent and Social Systems

The theory may help explain transitions in collective behavior, such as coordination breakdowns, group reorganizations, and social phase shifts.

(5) Formalization of Boundary Regions

Mathematical characterization of liminal regions—e.g., via manifold intersections or saddle geometry—could deepen the theoretical foundation.

6.5 Summary

The unified switching theory provides a structural framework that captures the essential features of switching phenomena across domains. By integrating insights from four distinct theoretical traditions, the theory offers a general account of how systems destabilize, traverse boundary regions, and reorganize into new stable modes. Although the theory is primarily structural and descriptive, it provides a powerful foundation for future mechanistic, quantitative, and empirical work.

Chapter 7. Conclusion

This paper has developed a unified theoretical framework for understanding switching phenomena across biological, cognitive, and organizational systems. Building on the analyses presented in Papers 1–4, we demonstrated that four distinct frameworks—SCAN, VSM, Constraint Geometry, and the 3D Minimal Model—share a common structural organization when applied to switching. Despite their differences in origin, terminology, and mathematical formulation, each framework describes switching as a process involving destabilization, expansion of degrees of freedom, traversal of a boundary region, geodesic transition, and re-stabilization.

By formalizing this shared structure, we introduced the **Unified Switching Theory**, which provides a general account of switching grounded in three core principles:

1. **Increase in task-irrelevant variability ($\text{Var}\perp$)**
2. **Boundary-region traversal (liminality)**
3. **Geodesic transition toward a new stable configuration**

This theory synthesizes structural, geometric, and dynamical perspectives into a coherent framework that explains why switching occurs, how it unfolds, and why its structure is preserved across domains.

The unified theory offers several key contributions. First, it provides a common language for describing switching phenomena that have traditionally been studied in isolation. Second, it clarifies the role of variability—particularly $\text{Var}\perp$ —as a universal indicator of destabilization and impending transition. Third, it identifies geodesic transitions as a general principle governing the movement between modes. Finally, it demonstrates the broad applicability of these principles through examples in locomotion, motor coordination, organizational restructuring, and cognitive mode switching.

Although the theory is primarily structural and descriptive, it lays the groundwork for future mechanistic and quantitative models. Further research is needed to refine the mathematical characterization of boundary regions, develop predictive models of $\text{Var}\perp$ dynamics, and test the theory empirically across diverse systems. The integration of this framework with robotics, machine learning, and multi-agent systems also represents a promising direction.

In summary, the unified switching theory provides a conceptual foundation for understanding mode transitions as a universal structural process. By revealing the deep commonalities among seemingly disparate systems, the theory opens new possibilities for cross-domain insights and advances our understanding of how complex systems reorganize, adapt, and evolve.

Appendix A. Notation and Mathematical Background

This appendix summarizes the notation and mathematical tools used throughout the paper.

A.1 State-Space Decomposition

The system state is decomposed into task-relevant (TR) and task-irrelevant (TI) components:

$$x = (x_{TR}, x_{TI}) \in \mathbb{R}^n.$$

Variability along the TI dimension is defined as:

$$\text{Var}\perp = \text{Var}(x_{TI}).$$

A.2 Stability and Attractor Geometry

Let $V(x)$ denote an effective potential or Lyapunov function. A stable mode corresponds to a local minimum of V :

$$\nabla V(x^*) = 0, \nabla^2 V(x^*) \succ 0.$$

Attractor shallowing corresponds to a reduction in curvature:

$$\nabla^2 V(x^*) \downarrow.$$

A.3 Feasible Regions and Optimal Faces

Constraint Geometry represents feasible solutions as a polytope $P \subset \mathbb{R}^n$. An optimal face $F \subseteq P$ satisfies:

$$F = \{x \in P \mid c^\top x = c^\top x^*\}.$$

Face expansion corresponds to:

$$F \subset F' \subset F''.$$

A.4 Geodesics

A geodesic $\gamma(t)$ minimizes path length:

$$\gamma = \arg \min_{\gamma} \int \|\dot{\gamma}(t)\| dt.$$

This concept underlies the unified theory's shortest-path transition principle.

Appendix B. Proof Sketches and Technical Details

This appendix provides formal justification for the structural equivalence and the unified switching principle.

B.1 Structural Equivalence Lemma

Lemma. If four frameworks each admit a decomposition into: (1) stable mode, (2) destabilization, (3) boundary region, (4) transition pathway, (5) re-stabilization, then they are structurally equivalent.

Proof Sketch. Each framework defines a mapping:

$$\Phi_i : \mathcal{S} \rightarrow \{1, 2, 3, 4, 5\}$$

assigning system states to stages. If all Φ_i share the same ordering and adjacency relations, then they induce isomorphic transition graphs. Thus, the frameworks are structurally equivalent. $\square$

B.2 Proof Sketch of the Unified Switching Principle

Sketch. Assume $Var_{\perp}$ increases monotonically. Then:

1. Stability decreases (by curvature reduction).
2. The feasible region expands (by face expansion).
3. The system enters a region where multiple minima or faces coexist.
4. The minimal transition path is a geodesic (by variational principles).
5. A new stable mode emerges (by local convexity).

Thus, the four conditions of the principle follow. $\square$

B.3 Saddle Geometry in the 3D Minimal Model

The saddle region is characterized by:

$$\nabla V = 0, \nabla^2 V \text{ has mixed signs.}$$

This corresponds to the liminal region in SCAN and the interior region in Constraint Geometry.

Appendix C. Dynamic Model Reproducibility

This appendix provides reproducibility details for the 3D minimal model used in this paper.

C.1 Model Equations

The minimal model is defined as:

$$\dot{x} = f(x, v)$$

with state:

$$x = (x_{TR}, x_{TI}, v).$$

C.2 Parameter Values

A representative parameter set is:

Parameter	Meaning	Value
k_{TR}	TR stiffness	1.2
k_{TI}	TI stiffness	0.3
d	damping	0.15
v_c	critical speed	2.1 m/s

C.3 Numerical Integration

Simulations use:

- 4th-order Runge–Kutta
- time step: 0.001 s
- initial conditions near the walking attractor

C.4 Transition Trajectories

The transition trajectory is identified by:

- saddle-connecting path
- maximal $\text{Var}\perp$
- minimal action integral

Appendix D. Static LP Example

This appendix provides a simple linear programming example illustrating optimal face expansion.

D.1 Problem Setup

$$\min c^\top x \text{ s.t. } Ax \leq b.$$

D.2 Face Expansion Under Parameter Change

Let $b \rightarrow b'$ with $b' > b$. Then the feasible region expands:

$$P \subset P'.$$

The optimal face expands accordingly:

$$F \subset F'.$$

D.3 Geodesic Movement Between Faces

The transition between F and F' follows a geodesic within P' .

Appendix E. Applications of the Unified Switching Theory

This appendix expands on the applications introduced in Chapter 5.

E.1 UCM Hypothesis

- TR/TI decomposition aligns naturally with $\text{Var} \perp$.
- Switching corresponds to TI variance expansion and manifold reorganization.

E.2 Organizational Mode Switching

- Hierarchical breakdown $\rightarrow$ liminal ambiguity $\rightarrow$ minimal restructuring $\rightarrow$ new hierarchy
- Direct mapping to the unified theory's five-stage structure

E.3 Bipedal Stance Switching

- Quiet stance $\rightarrow$ destabilization $\rightarrow$ boundary-region sway $\rightarrow$ geodesic shift $\rightarrow$ new stance mode
- Matches the unified switching sequence

References

- Aoi, S., Ohashi, T., Bamba, R., Fujiki, S., Tamura, D., Funato, T., Senda, K., Ivanenko, Y., & Tsuchiya, K. (2019). Neuromusculoskeletal model that walks and runs across a speed range with a few motor control parameter changes based on the muscle synergy hypothesis. *Scientific Reports*, 9, 707. <https://doi.org/10.1038/s41598-018-37460-3>
- Aoki, M. (1971). *Organizational and Planning Theory in Economics* [Soshiki to Keikaku no Keizai Riron]. Iwanami Shoten. (in Japanese).
- Arrow, K. J. (1974). *The Limits of Organization*. W. W. Norton & Company. (Japanese edition translated by Y. Murakami, Iwanami Shoten, 1976).
- Beer, S. (1966). *Decision and Control: The Meaning of Operational Research and Management Cybernetics*. John Wiley & Sons.
- Beer, S. (1972). *Brain of the Firm: A Development in Management Cybernetics*. Herder and Herder.

Diedrich, F. J., & Warren, W. H., Jr. (1995). Why change gaits? Dynamics of the walk-run transition. *Journal of Experimental Psychology: Human Perception and Performance*, 21(1), 183–202. <https://doi.org/10.1037/0096-1523.21.1.183>

Gordon, E. M., Chauvin, R. J., Van, A. N., Rajesh, A., Nielsen, A., Newbold, D. J., Lynch, C. J., Seider, N. A., Krimmel, S. R., Scheidter, K. M., Monk, J., Miller, R. L., Metoki, A., Montez, D. F., Zheng, A., Elbau, I., Madison, T., Nishino, T., Myers, M. J., Kaplan, S., Badke D'Andrea, C., Demeter, D. V., Feiglis, M., Ramirez, J. S. B., & Dosenbach, N. U. F. (2023). A somato-cognitive action network alternates with effector regions in motor cortex. *Nature*, 617, 351–359. <https://doi.org/10.1038/s41586-023-05964-2>

Kelso, J. A. S. (1995). *Dynamic Patterns: The Self-Organization of Brain and Behavior*. The MIT Press.

Nikaido, H. (1960). *Mathematical Methods of Modern Economics: Introduction to Topological Analysis* [Gendai Keizaigaku no Sugakuteki Hoho]. Iwanami Shoten. (in Japanese).

Nikaido, H. (1982). *Linear Mathematics for Economics* [Keizai no Tame no Senkei Sugaku] (New Mathematics Series 22). Baifukan. (in Japanese).

Obayashi, I., Aoi, S., Tsuchiya, K., & Kokubu, H. (2015). Common formation mechanism of basin of attraction for bipedal walking models by saddle hyperbolicity and hybrid dynamics. *Japan Journal of Industrial and Applied Mathematics*, 32(2), 315–332. <https://doi.org/10.1007/s13160-015-0173-z>

Scholz, J. P., & Schöner, G. (1999). The uncontrolled manifold concept: Identifying control variables for a functional task. *Experimental Brain Research*, 126(3), 289–306. <https://doi.org/10.1007/s002210050738>